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ON STABILIZATION OF DIFFERENTIAL SYSTEMS WITH HYBRID FEEDBACK CONTROL

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Abstract. In this paper two-dimensional systems of differential equations are considered together with their stabilization by a hybrid feedback control. A stabilizing hybrid control for an arbitrary controlled system that belongs to a certain category within two-dimensional systems is constructed as a result of this study and some stabilization proprieties of the system with the obtained hybrid control are presented.
Keywords: stabilization; hybrid feedback control; linear hybrid control; upper Lyapunov exponent

1. Notations

We will use the following notations: $C(\mathbb{R}^n)$ is the set of all continuous functions $u:[0,\infty)\to\mathbb{R}^n$, $C_s(\mathbb{R}^n)$ is the set of all piecewise continuous functions such that $u:[0,\infty)\to\mathbb{R}^n$, the euclidean norm $|\cdot|$ in the space $\mathbb{R}^n$ will be denoted by $|x|$, the set of all matrices with real entries of dimension $m\times n$ we denote by $M(m,n,\mathbb{R})$, $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ is the set of all linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $\sigma(A)$ is the set of all eigenvalues of a square matrix A, called the spectrum of A.

2. Formulation of the problem

Let us consider a controlled system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \tag{2.1}$$

where $x\in\mathbb{R}^n$ is the state vector, $y\in\mathbb{R}^m$ is the output vector, $u\in\mathbb{R}^\ell$ is the control vector. The system (2.1) is completely defined by the triple of matrices (A,B,C), where $A\in M(n,n,\mathbb{R})$, $B\in M(n,\ell,\mathbb{R})$ and $C\in M(m,n,\mathbb{R})$.

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In this paper we will consider the system (2.1) together with the so called hybrid feedback control. The notion of hybrid feedback control was given in several papers such as [1–3].

D e f i n i t i o n 2.1. A *hybrid automaton* is a set of six objects $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$, where

1. Q is a finite set of all the automaton's states;

2. I is a finite set called the input alphabet;

3. $M : Q \times I \to Q$ is an function that determines a new state of the automaton based on its previous state q and a element from the alphabet $i \in I$ that corresponds to the switching moment of the state;

4. $\mathcal{T} : Q \to (0, \infty)$ is a function that establishes the time period $\mathcal{T}(q)$ between two switching moments, satisfying $\inf\limits_{q \in Q} \mathcal{T}(q) > 0$;

5. $j : \mathbb{R}^m \to I$ is a function that corresponds to the output vector $y \in \mathbb{R}^m$ and the element $j(y)$ of I;

6. $q_0 = q(0)$ is the automaton's initial state.

Each hybrid automaton $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$ is associated to an operator $F_\Delta : P(\mathbb{R}^m) \to P(Q)$ called the *hybrid operator*. Such that $P(X)$ is a set of functions $v : [0, \infty) \to X$. Let us present the recursive definition of F_Δ.

D e f i n i t i o n 2.2. For any $y(\cdot) : [0, \infty) \to \mathbb{R}^m$, the function $q(\cdot) = (F_\Delta y)(\cdot) : [0, \infty) \to Q$ is defined by:

1. $q(0) = q_0$, $t_1 = \mathcal{T}(q_0)$, $q(t) = q_0$ $(\forall t \in [0, t_1))$;

2. $q(t_1) = M(q_0, j(y(t_1)))$, $t_2 = t_1 + \mathcal{T}(q(t_1))$, $q(t) = q(t_1)$, $(\forall t \in [t_1, t_2))$;

3. Let $k \in \{2, 3, \ldots\}$. Suppose that $t_0 = 0, t_1, \ldots, t_k$ and that the values of $q(t)$ for $t \in [0, t_k)$ were already defined. Then, t_{k+1} and $q(t)$ for $t \in [t_k, t_{k+1})$ are defined by:
$$q(t_k) = M(q(t_{k-1}),\ j(y(t_k))), \quad t_{k+1} = t_k + \mathcal{T}(q(t_k)), \quad q(t) = q(t_k) \quad (\forall t \in [t_k, t_{k+1})).$$

D e f i n i t i o n 2.3. A pair $u = (\Delta, \Phi)$, where $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$ is a hybrid automaton and $\Phi : \mathbb{R}^m \times Q \to \mathbb{R}^\ell$ is a function, is called *hybrid feedback control* (HFC).

The *hybrid control operator* $W_u : C(\mathbb{R}^m) \to C_s(\mathbb{R}^\ell)$, associated to the control $u = (\Delta, \Phi)$, is defined by
$$(W_u y)(t) = \Phi(y(t), (F_\Delta y)(t)), \qquad t \in [0, \infty),$$
where F_Δ is the operator that was recursively defined above.

R e m a r k 2.1. According to the Definition 2.3, the linear system (2.1) with the hybrid control $u = (\Delta, \Phi)$ is equivalent to a functional differential equation [4]

$$\dot{x}(t) = Ax(t) + B\Phi(Cx(t), (F_\Delta Cx)(t)), t \in [0, \infty). \tag{2.2}$$

D e f i n i t i o n 2.4. Let $u = (\Delta, \Phi)$ be a hybrid control of the system (2.1), where $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$.

The HFC u is called *linear hybrid control* (LHFC) if it satisfies the following conditions:

(a) the function $j : \mathbb{R}^m \to I$, satisfies the condition $j(\lambda y) = j(y)$ for any $y \in \mathbb{R}^m$ and $\lambda > 0$;

(b) the function $\Phi(y, q)$ is linear in relation to y.

We will denote the LHFC class by $\mathcal{LH} = \mathcal{LH}(\ell, m)$.

It is convenient to represent the LHFC u in the following manner : $u = (\Delta, \{G_q\}_{q\in Q})$, where $G_q \in M(\ell, m)$ $(q \in Q)$.

Therefore the *hybrid control operator* $W_u : C(\mathbb{R}^m) \to C_s(\mathbb{R}^\ell)$ associated with $u = (\Delta, \{G_q\}_{q\in Q})$ has the form of the following linear dependence:

$$(W_u y)(t) = G_{(F_\Delta y)(t)} y(t), t \in [0, \infty).$$

D e f i n i t i o n 2.5. Let (2.1) be a system with the triple $\Omega = (A, B, C)$ and with a control $u \in \mathcal{LH}$. The infimum of $\lambda \in \mathbb{R}$ with which for every solution of the system it holds:

$$|x(t)| \leq Me^{\lambda t}|x(0)|, \qquad t \in [0, \infty). \tag{2.3}$$

with M positive and independent from the solution constant is called *upper Lyapunov exponent* of the system (2.1) with the control u and is denoted by $\lambda(\Omega, u)$.

D e f i n i t i o n 2.6. *Upper exponent* of the system (2.1) with linear hybrid feedback control is the value $\lambda(\Omega, \mathcal{LH})$ defined by

$$\lambda(\Omega, \mathcal{LH}) = \inf_{u\in\mathcal{LH}} \lambda(\Omega, u).$$

Surely, the upper exponent is important because it characterizes the asymptotic behaviour of the solutions.

If the upper exponent $\lambda(\Omega, \mathcal{LH}) < 0$, then existis $u \in \mathcal{LH}$ such that the solution of the controllable system (2.1) exponentially stable which means that the system is stabilizable by LHFC.

It is clear, from the point of view of the stabilization of controllable systems, that it is good when $\lambda(\Omega, \mathcal{LH}) = -\infty$.

Consider the linear differential system with control:

$$\begin{cases} \dot{x} = A_\mu x + B_0 u \\ y = C_0 x \end{cases} \quad \text{with } \Omega_{[\mu]} = (A_\mu, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right) \tag{2.4}$$

this is, the system

$$\begin{cases} \dot{x}_1 = \mu x_1 + x_2 \\ \dot{x}_2 = -x_1 + \mu x_2 + u \\ y = x_1 \end{cases},$$

called the generalized harmonic oscillator. Note that the triple $\Omega_{[\mu]} = (A_\mu, B_0, C_0)$ of the system (2.4) is the canonical triple of the equivalence classes $H(2,0,\mu)$ where $\mu \in \{-1,0,1\}$. As in [3] and [5] we will not limit the study of the system to these three values of the parameter μ but will consider the system with an arbitrary parameter $\mu \in \mathbb{R}$.

We have categories of systems that can be stabilized by hybrid control and a hybrid control was already constructed for the canonical cases of these categories [3, 5]. Specifically, the category $H(2,0,\mu)$, which contains all the triples (A,B,C) that satisfy $BC = 0$, $CAB \neq 0$ will be examined. This category consists of three equivalence classes corresponding to cases when $\mu \in \{-1,0,1\}$ and the characteristic propriety of each of these classes is $CB = 0$, $CAB \neq 0$ and $\operatorname{sign} \operatorname{tr} A = \mu$, $\operatorname{tr} A = a_{11} + a_{22} + \cdots + a_{nn}$ is the trace of matrix A. The canonical form of these classes is

$$\Omega_{[\mu]} = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right).$$

In (2) and (6) a class of hybrid controls was presented. It stabilizes the system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases},$$

with the canonical triple $\Omega_{[\mu]}$.

Let $\Sigma = M(2,2,\mathbb{R}) \times (M(2,1,\mathbb{R}) \setminus \{O\}) \times (M(1,2,\mathbb{R}) \setminus \{O\})$, this means, Σ is the set of all the triples of matrices (A,B,C) where $A \in M(2,2,\mathbb{R})$, $B \in M(2,1,\mathbb{R})$ and $C \in M(1,2,\mathbb{R})$, so that B and C are non-zero matrices. Let us denote by $GL(2)$ the multiplicative group of the square non-singular real matrices of order 2.

D e f i n i t i o n 2.7. We define the applications $T_1(D)$, $T_2(m_1,m_2,m_3)$ and $T_3(\alpha)$ from Σ to Σ by the formulas:

$$\begin{aligned} &T_1(D)(A,B,C) = (DAD^{-1}, DB, CD^{-1}), \quad D \in GL(2); \\ &T_2(m_1,m_2,m_3)(A,B,C) = (m_1A, m_2B, m_3C), \\ &\qquad\qquad\qquad\qquad\qquad\qquad m_1 > 0,\ m_2, m_3 \in \mathbb{R} \setminus \{0\}; \\ &T_3(\alpha)(A,B,C) = (A + \alpha BC, B, C), \quad \alpha \in \mathbb{R}. \end{aligned}$$

Let us consider the set of all the applications defined above:

$$\begin{aligned} GT_0 = \{&T_1(D) \colon D \in GL(2)\} \cup \\ &\{T_2(m_1,m_2,m_3) \colon m_1 > 0;\ m_2, m_3 \in \mathbb{R} \setminus \{0\}\} \cup \{T_3(\alpha) \colon \alpha \in \mathbb{R}\}. \end{aligned}$$

It is clear that any element in $T \in GT_0$ is a bijective function $T : \Sigma \to \Sigma$, this means, is a transformation of the set Σ. Therefore, $GT_0 \subset B(\Sigma)$ where $B(\Sigma)$ is the group of all transformations on Σ with the binary operation that is the composition of transformations. In that way we defined the transformation's group GT, generated by the set GT_0.

By having an arbitrary triple Ω that satisfies $BC = 0$, $CAB \neq 0$ the goal is to construct a hybrid control with the triple Ω for the corresponding system, using the theorem from the next section. This means, to construct a hybrid control for an arbitrary system that belongs to the category in question. For that it is necessary to determine the parameters of the transformation T from GT so that $T(\Omega) = \Omega_{[\mu]}$ and with the aid on the inverse transformation T^{-1}, find the linear hybrid control that stabilizes the system Ω with any upper Lyapunov exponent.

This paper contains the solution for the problem described above. This is the main problem and the results presented are new.

3. Relation between hybrid trajectories of equivalent systems

Proposition 3.1. *Let the transformation $T \in GT$ be given and represented in the following form :*

$$T = T_1(D) \circ T_2(m_1, m_2, m_3) \circ T_3(\alpha)$$

for some matrix $D \in GL(2)$ and some constants $m_1 > 0$, $m_2, m_3 \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Then, the inverse transformation T^{-1} of T is defined by

$$T^{-1} = T_3(-\alpha) \circ T_2\left(m_1^{-1}, m_2^{-1}, m_3^{-1}\right) \circ T_1(D^{-1}).$$

Theorem 3.1. *Let the triples $\Omega_i = (A_i, B_i, C_i) \in \Sigma$ $(i = 1, 2)$ be given, such that $\Omega_2 = T(\Omega_1)$, $T \in GT$ can be written as:*

$$T = T_3(\alpha) \circ T_2(m_1, m_2, m_3) \circ T_1(D), \tag{3.1}$$

with some matrix $D \in GL(2)$ and some constants $m_1 > 0$, $m_2, m_3 \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$.

Let us consider two controllable systems (S_1) and (S_2) :

$$(S_1): \quad \begin{cases} \dot{x} = A_1 x + B_1 u \\ u = C_1 y \end{cases}, \qquad \begin{array}{l} \text{with hybrid control} \\ u_1 = (\Delta_1, \{\alpha_q^{(1)}\}_{q\in Q}) \in \mathcal{LH}(1,1), \\ \text{where } \Delta_1 = (Q, I, M, \mathcal{T}_1, j_1, q_0), \end{array}$$

$$(S_2): \quad \begin{cases} \dot{x} = A_2 x + B_2 u \\ u = C_2 y \end{cases}, \qquad \begin{array}{l} \text{with hybrid control} \\ u_2 = (\Delta_2, \{\alpha_q^{(2)}\}_{q\in Q})) \in \mathcal{LH}(1,1), \\ \text{where } \Delta_2 = (Q, I, M, \mathcal{T}_2, j_2, q_0), \end{array}$$

such that the components Q, I, M, q_0 of the hybrid automatons Δ_i are the same and

$$\begin{gathered} \mathcal{T}_2(q) = m_1^{-1}\mathcal{T}_1(q) \quad (\forall q \in Q), \quad j_2(y) = j_1(y \operatorname{sign} m_3) \quad (\forall y \in \mathbb{R}), \\ \alpha_q^{(2)} = \frac{m_1}{m_2 m_3}\alpha_q^{(1)} - \alpha \quad (\forall q \in Q). \end{gathered} \tag{3.2}$$

Consider the hybrid trajectories $h_i(t) = (x^{(i)}(t), q_i(t), \tau_i(t))$, $(t \in [0, \infty))$ of the systems (S_i) $(i = 1, 2)$, such that the initial conditions of the components $x^{(i)}$ of these trajectories satisfy the relation $x^{(2)}(0) = Dx^{(1)}(0)$. Then, the following relations take place: $\forall t \in [0, \infty)$

$$x^{(2)}(t) = Dx^{(1)}(m_1 t), \quad q_2(t) = q_1(m_1 t), \quad \tau_2(t) = m_1^{-1}\tau_1(m_1 t).$$

The results of the theorem above follow naturally from the results that are found in [2], however, some changes were necessary because of some inaccuracy found in it.

Corollary 3.1. *Let us consider the same systems with hybrid controls* (S_1) *and* (S_2) *as in Theorem 3.1. For any solution* $x^{(1)}$ *of the system* (S_1) *the exponential estimate is satisfied:*

$$|x^{(1)}(t)| \leq M_1\, e^{\lambda t}|x^{(1)}(0)|, \qquad t \in [0,\infty) \tag{3.3}$$

such that the constants $\lambda \in \mathbb{R}$ *and* $M_1 > 0$ *that do not depend on the solutions if and only if for any solution* $x^{(2)}$ *of system* (S_2) *the exponential estimate is satisfied:*

$$|x^{(2)}(t)| \leq M_2\, e^{m_1 \lambda t}|x^{(2)}(0)|, \qquad t \in [0,\infty) \tag{3.4}$$

such that $M_2 > 0$ *do not depend on the solution and the constant* $m_1 > 0$ *is the same as in the transformation* (3.1)*.*

P r o o f. By the Theorem 3.1, a function $x^{(1)} : [0,\infty) \to \mathbb{R}^2$ is a system's solution (S_1) if and only if the function $x^{(2)} : [0,\infty) \to \mathbb{R}^2$ defined by

$$x^{(2)}(t) = Dx^{(1)}(m_1 t), \qquad t \in [0,\infty),$$

which is the solution of the system (S_2). So, from the estimate (3.3) we have:

$$|x^{(2)}(t)| = |Dx^{(1)}(m_1 t)| \leq \|D\|\, |x^{(1)}(m_1 t)| \leq \|D\|\, M_1\, e^{m_1 \lambda t}|x^{(1)}(0)| =$$
$$\|D\|\, M_1\, e^{m_1 \lambda t}|D^{-1}x^{(2)}(0)| \leq M_2\, e^{m_1 \lambda t}|x^{(2)}(0)|, \qquad t \in [0,\infty)$$

where $M_2 = M_1\, \|D\|\, \|D^{-1}\|$. Reciprocally, from the estimate (3.4) we have:

$$|x^{(1)}(t)| = |D^{-1}x^{(2)}(m_1^{-1}t)| \leq \|D^{-1}\|\, |x^{(2)}(m_1^{-1}t)| \leq \|D^{-1}\|\, M_2\, e^{m_1 m_1^{-1} \lambda t}|x^{(2)}(0)|$$
$$= \|D^{-1}\|\, M_2\, e^{\lambda t}|Dx^{(1)}(0)| \leq M_1\, e^{\lambda t}|x^{(1)}(0)|, \qquad t \in [0,\infty)$$

where $M_1 = M_2\, \|D^{-1}\|\, \|D\|$.

Corollary 3.2. *Let us consider the same systems with the hybrid control* (S_1) *and* (S_2) *as in the Theorem 3.1, which means, the systems with the triples* $\Omega_i = (A_i, B_i, C_i)$ *such that* $\Omega_2 = T(\Omega_1)$ *where* T *is defined by* (3.1) *with controls* $u_i \in \mathcal{LH}$ *connected by* (3.2)*.Then the upper Lyapunov exponents of* (S_i) *satisfy the relation:*

$$\lambda(\Omega_2, u_2) = m_1\, \lambda(\Omega_1, u_1).$$

The corollary's 3.2 proof follows from the Corollary 3.1.

4. Transformation of the triple (A, B, C) in case $BC = 0$, $CAB \neq 0$ into canonical form

In this section the transformation $T \in GT$ will be determined in the form of a composition of the transformations $T_i(\cdot)$ $(i = 1, 2, 3)$ defined above that transform a triple Ω that satisfies $BC = 0$, $CAB \neq 0$, in the canonical triple

$$\Omega_{[\mu]} = (A_{[\mu]}, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right), \quad \mu \in \{-1, 0, 1\}. \tag{4.1}$$

Let the initial triple Ω be given and defined by

$$\Omega = (A, B, C) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_2 \;\; c_2] \right)$$

such that $CB = b_1c_1 + b_2c_2 = 0$, $CAB \neq 0$. Let $\mu = \operatorname{sign}(\operatorname{tr} A)$. According to the classification, there exists only one transformation $T \in GT$ such that $T(\Omega) = \Omega_{[\mu]}$. The goal now is to find the representation of this transformation T in terms of elements of matrices A, B and C. The problem is solved in some steps, described bellow.

1) First, the transformation $T_3(\beta)$ is applied, where

$$\beta = \begin{cases} \dfrac{2\det A - \operatorname{tr}^2 A}{2CAB} & \text{if } \operatorname{tr} A \neq 0 \\ \dfrac{\det A - 1}{CAB} & \text{if } \operatorname{tr} A = 0 \end{cases} = \frac{\det A - \frac{1}{2}\operatorname{tr}^2 A + |\mu| - 1}{CAB}. \tag{4.2}$$

We get a new triple

$$T_3(\beta)(\Omega) = T_3(\beta)(A, B, C) = (A + \beta BC, B, C) = (A_1, B_1, C_1) = \Omega_1.$$

As it can be noted, the only matrix that suffers some transformations is the matrix A, such that in the triple Ω_1 the matrices B_1 and C_1 are the same to the matrices B and C, respectively, from the initial triple Ω. Now the form of the matrix A_1 will be determined:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \beta \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} a_{11} + \beta b_1 c_1 & a_{12} + \beta b_1 c_2 \\ a_{21} + \beta b_2 c_1 & a_{22} + \beta b_2 c_2 \end{bmatrix}.$$

The goal of applying the transformation $T_3(\beta)$ is to obtain the matrix A_1 with two complex eigenvalues which have the same real and imaginary parts by modulo. More precisely, we have

$$\sigma(A_1) = \begin{cases} \left\{ \dfrac{\operatorname{tr} A}{2} - i \cdot \dfrac{\operatorname{tr} A}{2}, \dfrac{\operatorname{tr} A}{2} + i \cdot \dfrac{\operatorname{tr} A}{2} \right\}, & \text{if } \operatorname{tr} A \neq 0 \\ \{-i,\, i\}, & \text{if } \operatorname{tr} A = 0 \end{cases}$$

Note that the idea of using the transformation $T_3(\beta)$ with the described propriety of the spectrum of A_1 can be found in [6, p. 33], however, some changes were necessary due to some inaccuracy in the expressions of β and $\sigma(A_1)$.

2) Next, the transformation $T_2(\nu, 1, 1)$ is applied to the triple Ω_1 with

$$\nu = \begin{cases} \dfrac{2}{|\operatorname{tr} A|}, & \text{if } \mu \in \{-1, 1\} \\ 1, & \text{if } \mu = 0 \end{cases}. \tag{4.3}$$

The triple $\Omega_2 = (A_2, B_2, C_2) = T_2(\nu, 1, 1)(A_1, A_2, A_3)$ is obtained. Being that the two of the last parameters of T_2 are equal to 1, the matrices B and C remain the same. Thus, B_2 and C_2 are the same as B_1 and C_1, that are the matrices B and C from the initial triple Ω. The matrix A_2 has the following form:

$$A_2 = \nu A_1 = \begin{bmatrix} \nu(a_{11} + \beta b_1 c_1) & \nu(a_{12} + \beta b_1 c_2) \\ \nu(a_{21} + \beta b_2 c_1) & \nu(a_{22} + \beta b_2 c_2) \end{bmatrix}.$$

The goal of applying the given transformation $T_2(\nu, 1, 1)$ is to obtain the spectrum $\sigma(A_2) = \{\mu - i, \mu + i\}$ $(\forall \mu \in \{-1, 0, 1\})$.

3) The goal of this third step is to obtain the canonical matrix $A_{[\mu]}$, defined by (4.1) from the matrix A_2. This transformation was obtained from the theorem 9 in [7, p. 299].

Let us determine an eigenvector v of the matrix A_2 associated to the eigenvalue $\lambda = \mu + i$:

$$(A_2 - (\mu + i)I)v = 0 \quad \Rightarrow$$

$$\begin{cases} \big(\nu(a_{11} + \beta b_1 c_1) - (\mu + i)\big)\, v_1 + & \nu(a_{12} + \beta b_1 c_2)\, v_2 \ = 0 \\ \nu(a_{21} + \beta b_2 c_1)\, v_1 + & \big(\nu(a_{22} + \beta b_2 c_2) - (\mu + i)\big)\, v_2 \ = 0 \end{cases} \Rightarrow$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{\nu(a_{12} + \beta b_1 c_2)} \begin{bmatrix} \nu(a_{12} + \beta b_1 c_2) \\ \mu - \nu(a_{11} + \beta b_1 c_1) + i \end{bmatrix},$$

and define a real matrix V by

$$V = [\operatorname{Re} v \ \operatorname{Im} v] = \begin{bmatrix} 1 & 0 \\ \dfrac{\mu - \nu(a_{11} + \beta b_1 c_1)}{\nu(a_{12} + \beta b_1 c_2)} & \dfrac{1}{\nu(a_{12} + \beta b_1 c_2)} \end{bmatrix}.$$

Let us now apply the transformation $T_1(D)$ for the triple Ω_2 where

$$D = V^{-1} = \begin{bmatrix} 1 & 0 \\ \nu(a_{11} + \beta b_1 c_1) - \mu & \nu(a_{12} + \beta b_1 c_2) \end{bmatrix}. \tag{4.4}$$

We obtain the triple $\Omega_3 = (A_3, B_3, C_3) = T_1(D)(\Omega_2)$, such that, (see [7, p. 299]),

$$A_3 = DA_2D^{-1} = V^{-1}A_2V = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}.$$

Note that the matrices B_3 and C_3 are:

$$B_3 = DB = \begin{bmatrix} b_1 \\ \nu(a_{11}b_1 + a_{12}b_2) - \mu b_1 \end{bmatrix},$$

$$C_3 = CD^{-1} = CV = \begin{bmatrix} c_1 + \dfrac{c_2(\mu - \nu(a_{11} + \beta b_1 c_1))}{\nu(a_{12} + \beta b_1 c_2)} & \dfrac{c_2}{\nu(a_{12} + \beta b_1 c_2)} \end{bmatrix}.$$

So, by the steps 1), 2) and 3) the matrix $A_3 = A_{[\mu]}$ is obtained from the canonical triple $\Omega_{[\mu]}$. The goal of the next two steps in to find the transformations from the group GT that transform B_3 and C_3, to $B_0 = [0 \ \ 1]^\top$ and $C_0 = [1 \ \ 0]$, conserving the matrix $A_3 = A_{[\mu]}$.

4) As it was deducted in [6, p. 32], the matrix A_3 commutes with any matrix of form

$$L(\varphi, \varepsilon) = \begin{bmatrix} \varphi & \varepsilon \\ -\varepsilon & \varphi \end{bmatrix}$$

such that $L(\varphi, \varepsilon)A_3(L(\varphi, \varepsilon))^{-1} = A_3$. Let us now find the values of φ and ε such that $L(\varphi, \varepsilon)B_3 = B_0 = [0 \ \ 1]^\top$. Solving the linear system $L(\varphi, \varepsilon)B_3 = B_0$, this means

$$\begin{cases} b_1\, \varphi + & \big(\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1\big)\, \varepsilon \ = 0 \\ \big(\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1\big)\, \varphi - & b_1\, \varepsilon \ = 1 \end{cases},$$

in respect of φ and ε, we obtain

$$\varphi = \frac{\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2}, \quad \varepsilon = -\frac{b_1}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2}. \tag{4.5}$$

Let us now apply the transformation $T_1(L)$, where $L = L(\varphi, \varepsilon)$ with φ and ε defined by (4.5), that means

$$L = \frac{1}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2} \begin{bmatrix} \nu(a_{11}b_1 + a_{12}b_2) - \mu b_1 & -b_1 \\ b_1 & \nu(a_{11}b_1 + a_{12}b_2) - \mu b_1 \end{bmatrix}. \tag{4.6}$$

The triple $\Omega_4 = (A_4, B_4, C_4) = T_1(L)(\Omega_3)$ is obtained, where

$$A_4 = LA_3L^{-1} = A_3 = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \quad B_4 = LB_3 = B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_4 = C_3L^{-1} = \begin{bmatrix} \delta & 0 \end{bmatrix},$$

where

$$\delta = \Big(\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1\Big) \cdot \left(c_1 + \frac{c_2(\mu - \nu(a_{11} + \beta b_1 c_1))}{\nu(a_{12} + \beta b_1 c_2)}\right) - \frac{b_1 c_2}{\nu(a_{12} + \beta b_1 c_2)}.$$

Simplifying the expression of δ, according to (4.2), (4.3) and $CB = b_1c_1 + b_2c_2 = 0$, we obtain

$$\delta = \nu \cdot \det\,[B \;\; AB] \cdot \omega(B, C), \tag{4.7}$$

where

$$\omega(B, C) = \begin{cases} -\dfrac{c_1}{b_2}, & \text{if} \;\; b_2 \neq 0 \\ \dfrac{c_2}{b_1}, & \text{if} \;\; b_1 \neq 0. \end{cases}$$

Note that $-c_1/b_2 = c_2/b_1$ in case of $b_1b_2 \neq 0$, because $CB = 0$. The constant $\omega(B, C)$ has the following geometric interpretation: if consider B and $C^\top$ as vectors in $\mathbb{R}^2$, then we have $\omega(B, C) = |C^\top|/|B|$ if the angle between the vectors B and $C^\top$ are equal to $\pi/2$, and $\omega(B, C) = -|C^\top|/|B|$ if the angle between the vectors B and $C^\top$ is equal to $-\pi/2$.

5) At last, we apply the transformation $T_2(1, 1, \delta^{-1})$, obtaining the canonical triple $\Omega_{[\mu]}$ defined by (4.1).

6) Thus, a resultant transformation is presented:

$$T = T_2(1, 1, \delta^{-1}) \circ T_1(L) \circ T_1(D) \circ T_2(\nu, 1, 1) \circ T_3(\beta),$$

such that $T(\Omega) = \Omega_{[\mu]}$. By applying the propositions of the lemma 2.6 from the article [1], the transformation T can be presented in a much compact form:

$$T = T_1(LD) \circ T_2(\nu, 1, \delta^{-1}) \circ T_3(\beta),$$

such that the matrices L, D and the real constants ν, δ and β are defined in (4.6), (4.4), (4.3), (4.7) and (4.2), respectively. To conclude the formalization of T, we compute the matrix LD and simplify the expressions of its entries.

Thus, the following theorem has been proved:

Theorem 4.1. *Let be given a triple of matrices*

$$\Omega = (A, B, C) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_2 \ \ c_2] \right),$$

where $CB = 0$ *and* $CAB \neq 0$ *and the triple*

$$\Omega_{[\mu]} = (A_{[\mu]}, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \ \ 0] \right),$$

where $\mu = \operatorname{sign}(\operatorname{tr} A)$*. Therefore there exists a unique transformation* $T \in GT$ *such that* $T(\Omega) = \Omega_{[\mu]}$ *and that transformation can be represented as following:*

$$T = T_1(P) \circ T_2(\nu, 1, \delta^{-1}) \circ T_3(\beta),$$

where

$$\nu = \begin{cases} \dfrac{2}{|\operatorname{tr} A|}, & \text{if} \quad \mu \in \{-1, 1\} \\ 1, & \text{if} \quad \mu = 0 \end{cases}, \qquad \beta = \frac{\det A - \frac{1}{2}\operatorname{tr}^2 A + |\mu| - 1}{CAB},$$

$$\delta = \nu \cdot \det [B \ \ AB] \cdot \omega(B, C) \quad \textit{such that} \quad \omega(B, C) = \begin{cases} -\dfrac{c_1}{b_2}, & \text{if} \quad b_2 \neq 0 \\ \dfrac{c_2}{b_1}, & \text{if} \quad b_1 \neq 0 \end{cases}$$

and the elements of the matrix $P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$ *are defined by*

$$p_1 = \frac{\nu(a_{12}b_2 - \beta b_1^2 c_1)}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2},$$

$$p_2 = \frac{-b_1\nu(a_{12} + \beta b_1 c_2)}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2},$$

$$p_3 = \frac{b_1 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)(\nu(a_{11} + \beta b_1 c_1) - \mu)}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2},$$

$$p_4 = \frac{\nu(\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)(a_{12} + \beta b_1 c_2)}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2}.$$

Let us now present three examples of the triples $\Omega = (A, B, C) \in \Sigma$ from the category with the invariant $CB = 0$, $CAB \neq 0$ that belong to the three different equivalence classes $H(2, 0, \mu)$ for $\mu = 1$, $\mu = -1$ and $\mu = 0$, and construct for each of the triples, basing ourselves on the Theorem 4.1, the transformation T that maps this triple into the canonical triple $\Omega_{[\mu]}$.

E x a m p l e 4.1. Consider the triple of matrices

$$\Omega = (A, B, C) = \left(\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, [1 \ \ 4] \right).$$

Of course that $CB = 0$, $CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = \operatorname{sign} 4 = 1$. So, $\Omega \in H(2,0,1)$. Also note that $\sigma(A) = \{2-3i, 2+3i\}$. The transformation T that maps Ω to the canonical form

$$\Omega_{[1]} = \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right),$$

($T \in GT$, such that $T(\Omega) = (\Omega_{[1]})$) is defined by the formula:

$$T = T_1 \left(\begin{bmatrix} \frac{1}{37} & \frac{4}{37} \\ \frac{19}{74} & \frac{1}{37} \end{bmatrix} \right) \circ T_2 \left(\frac{1}{2}, 1, \frac{1}{37} \right) \circ T_3 \left(\frac{5}{74} \right).$$

Example 4.2. Let us consider the triple of matrices

$$\Omega = (A, B, C) = \left(\begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{5}{4} \end{bmatrix} \right),$$

$CB = 0$, $CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = \operatorname{sign}(-1) = -1$. Therefore $\Omega \in H(2,0,-1)$. Also note that $\sigma(A) = \{-4, 3\}$. The transformation T that maps Ω into a canonical form

$$\Omega_{[-1]} = \left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right),$$

is defined by:

$$T = T_1 \left(\begin{bmatrix} 0 & -\frac{1}{10} \\ -1 & \frac{3}{10} \end{bmatrix} \right) \circ T_2 \left(2, 1, -\frac{2}{25} \right) \circ T_3 (2).$$

Example 4.3. Consider the triple

$$\Omega = (A, B, C) = \left(\begin{bmatrix} -5 & -1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}, \begin{bmatrix} -6 & 2\sqrt{2} \end{bmatrix} \right).$$

$CB = 0, CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = \operatorname{sign} 0 = 0$. So, $\Omega \in H(2,0,0)$. Note, $\sigma(A) = \{-5, 5\}$. T that transforms Ω into the canonical form

$$\Omega_{[0]} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right),$$

this means, $T \in GT$ such that $T(\Omega) = (\Omega_{[0]})$, is defined by:

$$T = T_1 \left(\begin{bmatrix} -\frac{1}{3+10\sqrt{2}} & \frac{\sqrt{2}}{3(3+10\sqrt{2})} \\ \frac{5}{3+10\sqrt{2}} & \frac{91+15\sqrt{2}}{573} \end{bmatrix} \right) \circ T_2 \left(1, 1, \frac{1}{18+60\sqrt{2}} \right) \circ T_3 \left(-\frac{13}{9+30\sqrt{2}} \right).$$

5. Inverse Transformation

Let $\Omega = (A, B, C)$ be an arbitrary triple, such that $CB = 0$, $CAB \neq 0$. Having the transformation

$$T_d = T_1(P) \circ T_2(\nu, 1, \delta^{-1}) \circ T_3(\beta),$$

such that $T(\Omega) = \Omega_{[\mu]}$ where $\mu = \text{sign}\,(\text{tr}\,A)$ (see the Theorem 4.1), let us now determine the inverse transformation of T_d, this is, the transformation $T = T_d^{-1}$ such that $T(\Omega_{[\mu]}) = \Omega$.

According to the Proposition 3.1 the transformation T can be represented in the following form:

$$T = T_3(\alpha) \circ T_2(a, b, c) \circ T_1(D),$$

where

$$D = P^{-1}, \qquad a = \frac{1}{\nu}, \qquad b = 1, \qquad c = \delta, \qquad \alpha = -\beta.$$

Using the formulas of the Theorem 4.1, by rewriting the parameters of T in function of the matrices of the triple Ω, we get the following theorem:

Theorem 5.1. *Let the triple of matrices*

$$\Omega = (A, B, C) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_2 \ \ c_2] \right)$$

be given, where $CB = 0$, $CAB \neq 0$ *and the triple*

$$\Omega_{[\mu]} = (A_{[\mu]}, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \ \ 0] \right),$$

where $\mu = \text{sign}\,(\text{tr}\,A)$. *There exists a unique transformation* $T \in GT$ *such that* $T(\Omega_{[\mu]}) = \Omega$ *and that transformation can be represented in the following form:*

$$T = T_3(\alpha) \circ T_2(a, b, c) \circ T_1(D),$$

where

$$\alpha = \frac{\frac{1}{2}\text{tr}^{\,2} A - \det A + 1 - |\mu|}{CAB}, \quad a = \frac{|\text{tr}\,A|}{2} + 1 - |\mu|, \quad b = 1,$$

$$c = \frac{1}{a}\det\,[B \ AB]\,\omega(B, C) \ \text{ with } \ \omega(B, C) = \begin{cases} -\dfrac{c_1}{b_2}, & \text{if} \ \ b_2 \neq 0 \\ \dfrac{c_2}{b_1}, & \text{if} \ \ b_1 \neq 0 \end{cases}, \tag{5.1}$$

$$D = \begin{bmatrix} \dfrac{(a_{11} - a_{22})b_1 + 2a_{12}b_2}{2a} & b_1 \\ \dfrac{2a_{21}b_1 - (a_{11} - a_{22})b_2}{2a} & b_2 \end{bmatrix}.$$

For each triple from the examples 4.1, 4.2 and 4.3 let us present a transformation T that maps the canonical triple to these triples. The transformation T can be obtained from the Theorem 5.1 or by inverting the transformation that was obtained in each of the examples in the Section 4 with the use of the Proposition 3.1.

E x a m p l e 5.1. Consider the triple of matrices

$$\Omega = (A, B, C) = \left(\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, [1 \ \ 4] \right),$$

in which $CB = 0$, $CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = 1$. The transformation $T \in GT$ such that $T(\Omega_{[1]}) = \Omega$ is defined by the formula

$$T = T_1\left(\begin{bmatrix} -1 & 4 \\ \frac{19}{2} & -1 \end{bmatrix}\right) \circ T_2\,(2,1,37) \circ T_3\left(-\frac{5}{74}\right).$$

E x a m p l e 5.2. Consider the triple

$$\Omega = (A,B,C) = \left(\begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{5}{4} \end{bmatrix}\right),$$

such that $CB = 0$, $CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = -1$. The transformation $T \in GT$ such that $T(\Omega_{[-1]}) = \Omega$ is defined by the formula

$$T = T_1\left(\begin{bmatrix} -3 & -1 \\ -10 & 0 \end{bmatrix}\right) \circ T_2\left(\frac{1}{2}, 1, -\frac{25}{2}\right) \circ T_3\,(-2).$$

E x a m p l e 5.3. Consider the triple

$$\Omega = (A,B,C) = \left(\begin{bmatrix} -5 & -1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}, \begin{bmatrix} -6 & 2\sqrt{2} \end{bmatrix}\right),$$

such that $CB = 0$, $CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = 0$. The transformation $T \in GT$ such that $T(\Omega_{[0]}) = \Omega$ is defined by the formula

$$T = T_1\left(\begin{bmatrix} -3-5\sqrt{2} & \sqrt{2} \\ 15 & 3 \end{bmatrix}\right) \circ T_2\left(1, 1, 18+60\sqrt{2}\right) \circ T_3\left(\frac{13}{9+30\sqrt{2}}\right).$$

6. Construction of a stabilizing hybrid control for case $CB = 0$, $CAB \neq 0$

Consider the controllable differential linear two-dimensional system:

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1u \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_2u \\ y = c_1x_1 + c_2x_2 \end{cases} \tag{6.1}$$

where $u(\cdot) : [0,\infty) \to \mathbb{R}$ depends only from the output $u(\cdot) : [0,\infty) \to \mathbb{R}$ by a linear hybrid control. Suppose that the real parameters a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, c_1, c_2 of the system that satisfy the conditions:

$$b_1c_1 + b_2c_2 = 0, \qquad a_{11}b_1c_1 + a_{12}b_2c_1 + a_{21}b_1c_2 + a_{22}b_2c_2 \neq 0. \tag{6.2}$$

This section contains the main results of this paper: the control $u \in \mathcal{LH}$ that stabilizes the system (6.1), satisfying (6.2), such that the solution's norm decreases exponentially with any Lyapunov exponent.

Note that the system (6.1) with the conditions (6.2) in its vectorial form is:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{6.3}$$

in which the triple of matrices

$$\Omega = (A, B, C) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_2 \;\; c_2] \right)$$

satisfies $CB = 0$ and $CAB \neq 0$. Thus, we have the triple from the class $H(2, 0, \mu)$ where $\mu = \operatorname{sign}(\operatorname{tr} A) \in \{-1, 0, 1\}$. The canonical form of the class $H(2, 0, \mu)$ is

$$\Omega_{[\mu]} = (A_{[\mu]}, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right).$$

According to the Theorem 5.1 the transformation $T \in GT$ exists and is unique and $T(\Omega_{[\mu]}) = \Omega$. This transformation can be presented as following:

$$T = T_3(\alpha) \circ T_2(a, b, c) \circ T_1(D) \tag{6.4}$$

such that the constants α, a, b, c and the matrix D are defined by the formulas (5.1).

Let us generalize the results concerning the stabilization of the system $\Omega_{[\mu]}$ by a control $\mathcal{A}(R, \delta, m) \in \mathcal{LH}$, for the system with an arbitrary triple Ω such that $CB = 0$, $CAB \neq 0$. The generalization is based on the theorems 3.1 and 5.1.

Firstly, let us define the LHFC $\mathcal{H}(\Omega, R, \delta, m) \in \mathcal{LH}$ such that $R > 0$, $\delta > 0$ and $m \in \{0, 1\}$ in the following way. If (S_1) is the system with the triple $\Omega_{[\mu]}$ and control $u_1 = \mathcal{A}(R, \delta, m)$ and (S_2) is the system with the triple Ω and control $u_2 = \mathcal{H}(\Omega, R, \delta, m)$, then the parameters of the control u_2 can be expressed by the parameters of the control u_1 using the formulas (3.2) from the Theorem 3.1 with the use of the expressions (5.1) from the Theorem 5.1 for the transformation parameters T (T has the form (6.4) such that $T(\Omega_{[\mu]}) = \Omega$).

D e f i n i t i o n 6.1. Given $\Omega \in \Sigma$ defined by (6.3) where $CB = 0$ and $CAB \neq 0$ and given $R > 0$, $\delta > 0$ and $m \in \{0, 1\}$ the LHFC $\mathcal{H}(\Omega, R, \delta, m) \in \mathcal{LH}$ is defined by $\mathcal{H}(\Omega, R, \delta, m) = (\Delta, \{\alpha_q\}_{q \in Q})$ where the components of the hybrid automaton $\Delta = (Q, I, M, T, j, q_0)$ are given by

$$\begin{aligned} & Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\}, \\ & M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d, \\ & \mathcal{T}(q_d) = \mathcal{T}_d(R, a) = \frac{3\pi}{2a\sqrt{1+R}}, \quad \mathcal{T}(q_-) = \delta, \\ & j(y) = \begin{cases} i_+ & \text{if} \;\; \nu y \geq 0 \\ i_- & \text{if} \;\; \nu y < 0 \end{cases}, \qquad q_0 = \begin{cases} q_- & \text{if} \;\; m = 0 \\ q_d & \text{if} \;\; m = 1 \end{cases} \end{aligned} \tag{6.5}$$

such that

$$\begin{aligned} & a = \frac{|\operatorname{tr} A|}{2} + 1 - |\mu|, \quad \text{where} \quad \mu = \operatorname{sign}(\operatorname{tr} A), \\ & c = \frac{1}{a} \det[B \; AB]\, \omega(B, C) \quad \text{where} \quad \omega(B, C) = \begin{cases} -\dfrac{c_1}{b_2}, & \text{if} \;\; b_2 \neq 0 \\ \dfrac{c_2}{b_1}, & \text{if} \;\; b_1 \neq 0 \end{cases}, \\ & \alpha = \frac{\frac{1}{2}\operatorname{tr}^2 A - \det A + 1 - |\mu|}{CAB}, \quad \nu = \operatorname{sign}(c), \end{aligned} \tag{6.6}$$

and $\{\alpha_q\}_{q\in Q} = \{\alpha_{q_-}, \alpha_{q_d}\}$ where $\alpha_{q_-} = 0$ and $\alpha_{q_d} = -\left(\frac{a}{c}R + \alpha\right)$.

The families of hybrid controls are introduced:

$$\mathcal{H}(\Omega, R) = \left\{\mathcal{H}(\Omega, R, \delta, m)\colon 0 < \delta < \frac{\pi}{4a\sqrt{1+R}},\ m \in \{0,1\}\right\} \quad (R > 0),$$

$$\mathcal{H}(\Omega) = \bigcup_{R>0} \mathcal{H}(\Omega, R).$$

It is clear that $\mathcal{H}(\Omega, R) \subset \mathcal{H}(\Omega) \subset \mathcal{LH}$.

We define the function $\Lambda : (0,\infty) \to (0,\infty)$ by

$$\Lambda(R) = \frac{\sqrt{1+R}\ln(1+R)}{\pi\left(3+\sqrt{1+R}\right)}. \tag{6.7}$$

We remember that in this section we always consider the system (6.1) satisfying the conditions (6.2), or, indeed, the system (6.3) with triple $\Omega = (A,B,C)$ satisfying the condition $CB = 0$, $CAB \neq 0$. For convenience we designate this system for (S).

From the Corollary 2, Theorem 3 and the main results about the stabilization of the system with the triple Ω_μ from the papers (2) and (6) we get the main result of this paper which is stated in the following theorem:

Theorem 6.1. *For any* $R > 0$, $\lambda(\Omega, \mathcal{H}(\Omega, R)) = a(\mu - \Lambda(R))$, *where* μ *and* a *are defined in* (6.6).

Theorem 6.2. *We have the following statements:*

1. *If* $\operatorname{tr} A \le 0$ *(this is, when* $\mu = -1$ *ou* $\mu = 0$*), then* $\forall R > 0$ *the system* (S) *is stabilizable by a family of hybrid controls* $\mathcal{H}(\Omega, R)$.
2. *If* $\operatorname{tr} A > 0$ *(this is, when* $\mu = 1$*), then in case* $R > \Lambda^{-1}(1)$, *the system* (S) *is stabilizable by a family of hybrid controls* $\mathcal{H}(\Omega, R)$ *and in case* $R < \Lambda^{-1}(1)$ *the system* (S) *is not stabilizable by a family of hybrid controls* $\mathcal{H}(\Omega, R)$.

Theorem 6.3. *For any* $\Omega \in \Sigma$, $CB = 0$, $CAB \neq 0$ *it is valid that* $\lambda(\Omega, \mathcal{H}(\Omega)) = -\infty$.

R e m a r k 6.1. According to the Theorem 6.3, the system (S) is stabilizable by the hybrid controls from the family $\mathcal{H}(\Omega)$, such that the negative upper Lyapunov exponent in the solution estimate can be as large by modulo as we define it.

Let us complement the theorems 6.1–6.3 with a result which wording is more convenient for the applications. Also note that exists $\Lambda^{-1} : (0,\infty) \to (0,\infty)$ such that $\lim\limits_{s\to 0+} \Lambda^{-1}(s) = 0$ and $\lim\limits_{s\to+\infty} \Lambda^{-1}(s) = +\infty$. For convenience, let us extend the function Λ^{-1} to any set $\mathbb{R}$ by assigning, by definition $\Lambda^{-1}(s) = 0$ when $s \le 0$.

Theorem 6.4. *Let* $N > 0$ *be an arbitrary constant. Then, for any positive number* R *that satisfies*

$$R > \Lambda^{-1}\left(\operatorname{sign}(\operatorname{tr} A) + N\left(\frac{|\operatorname{tr} A|}{2} + 1 - |\mu|\right)^{-1}\right) \tag{6.8}$$

where $\Lambda(R)$ *is defined in* (6.7)*, exists a* $\delta_0 = \delta_0(\mathrm{tr}\, A, N, R) > 0$ *such that* $\forall \delta \in (0, \delta_0)$ *and* $\forall m \in \{0, 1\}$ *any solution of the equation* $x : [0, \infty) \to \mathbb{R}^2$ *of the system* (S) *with LHFC* $\mathcal{H}(\Omega, R, \delta, m)$ *satisfies the exponential estimate*

$$|x(t)| \le M\, e^{-Nt} |x(0)|, \qquad t \in [0, \infty)$$

where the constant $M = M(\Omega, R, \delta, m) > 0$ *does not depend on the solution* $x(\cdot)$*.*

7. Examples of the systems that satisfy $CB = 0$, $CAB \neq 0$ and stabilizing hybrid controls

In this section we will consider three specific systems of type (6.1) that correspond to the triples (A, B, C) considered in the examples from the sections 4 and 5. For these systems, based on the results of the Section 6, linear hybrid controls that stabilize it will be presented. Even more, the chosen control parameters are the ones that decrease the solution's norm as in (6.4) with a given upper Lyapunov exponent $-N$. For convenience, consider that the function Λ^{-1} is prolonged to $\mathbb{R}$ where by definition, $\Lambda^{-1}(s) = 0$ when $s \le 0$.

E x a m p l e 7.1. Consider the system:

$$\begin{cases} \dot{x}_1 = x_1 - 2x_2 + 4u \\ \dot{x}_2 = 5x_1 + 3x_2 - u \\ y = x_1 + 4x_2 \end{cases} \tag{7.1}$$

or, in the vectorial form:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \text{with} \quad \Omega = (A, B, C) = \left(\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, [1 \;\; 4] \right). \tag{7.2}$$

We have $CB = 0$ and $CAB \neq 0$. Let us compute the constants μ, a, c, ν, α by the formulas (6.6):

$$\begin{gathered} \mu = \mathrm{sign}\,(\mathrm{tr}\, A) = 1, \quad a = \frac{|\mathrm{tr}\, A|}{2} + 1 - |\mu| = 2, \quad c = \frac{c_2}{ab_1} \det\,[B \;\; AB] = 37, \\ \nu = \mathrm{sign}\,(c) = 1, \quad \alpha = \frac{\frac{1}{2}\mathrm{tr}^{\,2} A - \det A + 1 - |\mu|}{CAB} = -\frac{5}{74}. \end{gathered} \tag{7.3}$$

Consider the hybrid control $\mathcal{H}(\Omega, R, \delta, m) = ((Q, I, M, T, j, q_0), \{\alpha_-, \alpha_d\}) \in \mathcal{LH}$ defined in the Section 6. According to the Definition 6.1 and the expressions (7.3), the control components are given by:

$$Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\},$$

$$M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d,$$

$$\mathcal{T}(q_d) = \mathcal{T}_d(R, a) = \frac{3\pi}{4\sqrt{1 + R}}, \quad \mathcal{T}(q_-) = \delta,$$

$$j(y) = \begin{cases} i_+ & \text{if} \;\; y \ge 0 \\ i_- & \text{if} \;\; y < 0 \end{cases}, \qquad q_0 = \begin{cases} q_- & \text{if} \;\; m = 0 \\ q_d & \text{if} \;\; m = 1 \end{cases},$$

$$\alpha_{q_-} = 0, \qquad \alpha_{q_d} = \frac{1}{74}(5 - 4R).$$

The theorems 6.2 and 6.4 imply the following conclusions about the system (7.1) with linear hybrid control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 1. For any $m \in \{0,1\}$, $R > \Lambda^{-1}(1) \approx 69.89$ where Λ is defined in (6.7) and for all sufficiently small $\delta > 0$ the system (7.1) is stabilizable by the hybrid control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 2. Let $N > 0$. For any $m \in \{0,1\}$, $R > \Lambda^{-1}(1 + N/2)$ and for all sufficiently small $\delta > 0$, any solution $x : [0,\infty) \to \mathbb{R}^2$ of the system (7.1) with control $\mathcal{H}(\Omega, R, \delta, m)$ satisfies the exponential estimate

$$|x(t)| \le M\, e^{-Nt}|x(0)|, \qquad t \in [0,\infty),$$

where the constant $M = M(\Omega, R, \delta, m) > 0$ does not depend on the solution $x(\cdot)$.

For example, if a decrease of the solution with $N = 2$ is needed, then we can conclude that if $R > \Lambda^{-1}(2) \approx 977.35$ and $\delta > 0$ is sufficiently small, then any solution x of the system (7.1) with control $\mathcal{H}(\Omega, R, \delta, 0)$ or $\mathcal{H}(\Omega, R, \delta, 1)$ satisfies the condition

$$|x(t)| \le M\, e^{-2t}|x(0)|, \qquad t \in [0,\infty)$$

where $M > 0$ does not depend on the solution.

E x a m p l e 7.2. Consider the system:

$$\begin{cases} \dot{x}_1 = x_1 + 2x_2 - u \\ \dot{x}_2 = 5x_1 - 2x_2 \\ y = \frac{5}{4}x_2 \end{cases} \tag{7.4}$$

or, in its vectorial form:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \text{with} \quad \Omega = (A,B,C) = \left(\begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{5}{4} \end{bmatrix} \right). \tag{7.5}$$

We have $CB = 0$ and $CAB \ne 0$. Let us compute the constants μ, a, c, ν, α by the formulas (6.6):

$$\begin{gathered} \mu = \operatorname{sign}(\operatorname{tr} A) = -1, \quad a = \frac{|\operatorname{tr} A|}{2} + 1 - |\mu| = \frac{1}{2}, \quad c = \frac{c_2}{ab_1}\det[B\ \ AB] = -\frac{25}{2}, \\ \nu = \operatorname{sign}(c) = -1, \qquad \alpha = \frac{\frac{1}{2}(\operatorname{tr} A)^2 - \det A + 1 - |\mu|}{CAB} = -2. \end{gathered} \tag{7.6}$$

Consider the hybrid control $\mathcal{H}(\Omega, R, \delta, m) = ((Q, I, M, T, j, q_0), \{\alpha_-, \alpha_d\}) \in \mathcal{LH}_2$ defined in the Section 6. According to the Definition 6.1 and the expressions (7.6), the components of this control are given by:

$$Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\},$$
$$M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d,$$
$$\mathcal{T}(q_d) = \mathcal{T}_d(R, a) = \frac{3\pi}{\sqrt{1+R}}, \quad \mathcal{T}(q_-) = \delta,$$
$$j(y) = \begin{cases} i_+ & \text{if } \ y \le 0 \\ i_- & \text{if } \ y > 0 \end{cases}, \qquad q_0 = \begin{cases} q_- & \text{if } \ m = 0 \\ q_d & \text{if } \ m = 1 \end{cases},$$
$$\alpha_{q_-} = 0, \qquad \alpha_{q_d} = \frac{R}{25} + 2.$$

Theorems 6.2 and 6.4 imply the following about the system (7.4) with linear hybrid feedback control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 1. For any $m \in \{0,1\}$, $R > \Lambda^{-1}(1) \approx 69.89$ where Λ is defined in (6.7) and for all sufficiently small $\delta > 0$ the system (7.4) is stabilizable by the hybrid control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 2. Let $N > 0$. For any $m \in \{0,1\}$, $R > \Lambda^{-1}(2N-1)$ and for all small $\delta > 0$, any solution $x : [0,\infty) \to \mathbb{R}^2$ of the system (7.4) with control $\mathcal{H}(\Omega, R, \delta, m)$ satisfies the exponential estimate

$$|x(t)| \le M\, e^{-Nt}|x(0)|, \qquad t \in [0,\infty),$$

where the constant $M = M(\Omega, R, \delta, m) > 0$ does not depend on the solution $x(\cdot)$.

For example, if a decrease of the solution with $N = 2$ is needed, then we can conclude that if $R > \Lambda^{-1}(3) \approx 15545$ and $\delta > 0$ is sufficiently small, then any solution x of the system (7.4) with control $\mathcal{H}(\Omega, R, \delta, 0)$ or $\mathcal{H}(\Omega, R, \delta, 1)$ satisfies the condition

$$|x(t)| \le M\, e^{-2t}|x(0)|, \qquad t \in [0,\infty)$$

where $M > 0$ does not depend on the solution.

E x a m p l e 7.3. Consider the system

$$\begin{cases} \dot{x}_1 = -5x_1 - x_2 + \sqrt{2}\,u \\ \dot{x}_2 = 5x_2 - 2x_2 + 3u \\ y = -6x_1 + 2\sqrt{2}\,x_2 \end{cases} \tag{7.7}$$

and in its vectorial form

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \text{with } \Omega = (A,B,C) = \left(\begin{bmatrix} -5 & -1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}, \begin{bmatrix} -6 & 2\sqrt{2} \end{bmatrix} \right), \tag{7.8}$$

$CB = 0$ and $CAB \ne 0$. Let us compute the constants μ, a, c, ν, α by the formulas (6.6):

$$\begin{gathered} \mu = \operatorname{sign}(\operatorname{tr} A) = 0, \ a = \frac{|\operatorname{tr} A|}{2} + 1 - |\mu| = 1, \ c = \frac{c_2}{ab_1}\det\,[B \ \ AB] = 18 + 60\sqrt{2}, \\ \nu = \operatorname{sign}(c) = 1, \qquad \alpha = \frac{\frac{1}{2}\operatorname{tr}^2 A - \det A + 1 - |\mu|}{CAB} = \frac{13}{9 + 30\sqrt{2}}. \end{gathered} \tag{7.9}$$

Consider the hybrid control $\mathcal{H}(\Omega, R, \delta, m) = ((Q, I, M, T, j, q_0), \{\alpha_-, \alpha_d\}) \in \mathcal{LH}_2$ defined in the Section 6. According to the Definition 6.1 and the expressions (7.9), the components of this control are given by:

$$Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\},$$

$$M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d,$$

$$\mathcal{T}(q_d) = \mathcal{T}_d(R, a) = \frac{3\pi}{2\sqrt{1+R}}, \quad \mathcal{T}(q_-) = \delta,$$

$$j(y) = \begin{cases} i_+ & \text{if} \quad y \geq 0 \\ i_- & \text{if} \quad y < 0 \end{cases}, \qquad q_0 = \begin{cases} q_- & \text{if} \quad m = 0 \\ q_d & \text{if} \quad m = 1 \end{cases},$$

$$\alpha_{q_-} = 0, \qquad \alpha_{q_d} = -\frac{R + 26}{6(3 + 10\sqrt{2})}.$$

Theorems 6.2 and 6.4 imply the following about the system (7.7) with linear hybrid feedback control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 1. For any $m \in \{0, 1\}$, $R > \Lambda^{-1}(1) \approx 69.89$ where Λ is defined in (6.7) and for all sufficiently small $\delta > 0$ the system (7.7) is stabilizable by the hybrid control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 2. Let $N > 0$. For any $m \in \{0, 1\}$, $R > \Lambda^{-1}(N)$ for all small $\delta > 0$ any solution $x : [0, \infty) \to \mathbb{R}^2$ of the system (7.7) with control $\mathcal{H}(\Omega, R, \delta, m)$ satisfies the exponential estimate

$$|x(t)| \leq M\, e^{-Nt}|x(0)|, \qquad t \in [0, \infty),$$

where the constant $M = M(\Omega, R, \delta, m) > 0$ does not depend on the solution $x(\cdot)$.

For example, if a decrease of the solution with $N = 2$ is needed, then we can conclude that if $R > \Lambda^{-1}(2) \approx 977.35$ and $\delta > 0$ is sufficiently small, then any solution x of the system (7.7) with control $\mathcal{H}(\Omega, R, \delta, 0)$ or $\mathcal{H}(\Omega, R, \delta, 1)$ satisfies the condition

$$|x(t)| \leq M\, e^{-2t}|x(0)|, \qquad t \in [0, \infty)$$

where $M > 0$ does not depend on the solution.

REFERENCES

1. Litsyn E., Nepomnyashchikh Y., Ponossov A. Classification of linear dynamical systems in the plane admitting a stabilizing hybrid feedback control. *Journal on Dynamical and Control Systems*, 2000, vol. 6, no. 4, pp. 477-501.

2. Litsyn E., Myasnikova M., Nepomnyashchikh Y., Ponossov A. Efficient criteria for stabilization of planar linear systems by hybrid feedback controls. *Abstract and Applied Analysis*, 2004, no. 6, pp. 487-499.

3. Litsyn E., Nepomnyashchikh Y., Ponossov A. Control of the Lyapunov exponents of dynamical systems in the plane with incomplete observation. *Nonlinear Analysis. Series A: Theory, Methods & Applications, Special issue: Hybrid Systems and Applications*, 2005, vol. 64, no. 2, pp. 329-351.

4. Alves M.J. *Singulyarnye funktsional'no-differentsial'nye uravneniya vtorogo poryadka* [Singular Second Order Functional Differential Equations]. Perm, Perm State University Publ., 2000, 179 p. (In Russian).

5. Korotaeva I.G. *Ehksponentsial'naya stabilizatsiya dvumernykh gibridnykh system* [Exponential Stabilization of Two-Dimensional Linear Hybrid Systems]. Perm, 2003, 35 p. (In Russian).

6. Myasnikova M.A. *Klassifikatsiya lineynykh dvumernykh gibridnykh system* [Classification of Two-Dimensional Linear Hybrid Systems]. Perm, 2003, 51 p. (In Russian).

7. Lay D.C. *Linear Algebra and Its Applications.* Boston, Pearson Education Inc., 2012, 472 p.

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О СТАБИЛИЗАЦИИ ДИФФЕРЕНЦИАЛЬНЫХ ГИБРИДНЫХ УПРАВЛЯЕМЫХ СИСТЕМ С ОБРАТНОЙ СВЯЗЬЮ

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Аннотация. В данной статье рассматриваются двумерные системы дифференциальных уравнений со стабилизирующим гибридным управлением с помощью обратной связи. В результате исследования для произвольной системы управления, принадлежащей определенному классу двумерных систем, построено стабилизирующее гибридное управление и представлены некоторые стабилизирующие свойства системы с полученным гибридным управлением.
Ключевые слова: стабилизация; управление гибридной обратной связью; линейное гибридное управление; верхний показатель Ляпунова

СПИСОК ЛИТЕРАТУРЫ

1. *Litsyn E., Nepomnyashchikh Y., Ponossov A.* Classification of linear dynamical systems in the plane admitting a stabilizing hybrid feedback control // Journal on Dynamical and Control Systems. 2000. Vol. 6. № 4. P. 477-501.

2. *Litsyn E., Myasnikova M., Nepomnyashchikh Y., Ponossov A.* Efficient criteria for stabilization of planar linear systems by hybrid feedback controls // Abstract and Applied Analysis. 2004. № 6. P. 487-499.

3. *Litsyn E., Nepomnyashchikh Y., Ponossov A.* Control of the Lyapunov exponents of dynamical systems in the plane with incomplete observation // Nonlinear Analysis. Series A: Theory, Methods & Applications. Special issue: Hybrid Systems and Applications. 2005. Vol. 64. № 2. P. 329–351.

4. *Alves M.J.* Сингулярные функционально-дифференциальные уравнения второго порядка. Пермь: Изд-во Перм. гос. ун-та, 2000. 179 с.

5. *Коротаева И.Г.* Экспоненциальная стабилизация двумерных гибридных систем. Пермь, 2003. 35 с.

6. *Мясникова М.А.* Классификация линейных двумерных гибридных систем. Пермь, 2003. 51 с.

7. *Lay D.C.* Linear Algebra and Its Applications. Boston: Pearson Education Inc., 2012. 472 p.

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